TRAIN TRACKS AND THE GROMOV BOUNDARY OF THE COMPLEX OF CURVES

URSULA HAMENSTÄDT

1. INTRODUCTION

Consider a compact oriented surface S of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We require that $3g - 3 + m \geq 2$; this rules out a sphere with at most 4 punctures and a torus with at most one puncture.

In [Ha], Harvey defined the *complex of curves* $\mathcal{C}(S)$ for S. The vertices of this complex are free homotopy classes of simple closed curves on S. The simplices in $\mathcal{C}(S)$ are spanned by collections of such curves which can be realized disjointly. Thus the dimension of $\mathcal{C}(S)$ equals $3g - 3 + m - 1$ (recall that $3g - 3 + m$ is the number of curves in a *pants decomposition* of S).

The *extended mapping class group* $\tilde{\mathcal{M}}_{g,m}$ of S is the group of all isotopy classes of homeomorphisms of S. It acts naturally on the complex of curves as a group of simplicial automorphisms. Even more is true: If S is not a torus with 2 punctures then the extended mapping class group is *precisely* the group of simplicial automorphisms of $\mathcal{C}(S)$ (see [I] for references and a sketch of the proof).

Providing each simplex in $\mathcal{C}(S)$ with the standard euclidean metric of side-length 1 equips the complex of curves with the structure of a geodesic metric space whose isometry group is just $\tilde{\mathcal{M}}_{g,m}$ (except for the twice punctured torus). However, this metric space is not locally compact. Masur and Minsky [MM1] showed that nevertheless the geometry of $\mathcal{C}(S)$ can be understood quite explicitly. Namely, $\mathcal{C}(S)$ is hyperbolic of infinite diameter. Recall that for some $\delta > 0$ a geodesic metric space is δ-*hyperbolic in the sense of Gromov* if it satisfies the δ-*thin triangle condition*: For every geodesic triangle with sides a, b, c the side c is contained in the δ-neighborhood of $a \cup b$. Later Bowditch [B] gave a simplified proof of the result of Masur and Minsky which can also be used to compute explicit bounds for the hyperbolicity constant δ.

A δ-hyperbolic geodesic metric space X admits a *Gromov boundary* which is defined as follows. Fix a point $p \in X$ and for two points $x, y \in X$ define the *Gromov product* $(x, y)_p = \frac{1}{2}(d(x, p) + d(y, p) - d(x, y))$. Call a sequence $(x_i) \subset X$ *admissible* if $(x_i, x_j)_p \to \infty$ $(i, j \to \infty)$. We define two admissible sequences $(x_i), (y_i) \subset X$ to be *equivalent* if $(x_i, y_i)_p \to \infty$. Since X is hyperbolic, this defines indeed an equivalence relation (see [BH]). The Gromov boundary ∂X of X is then

the set of equivalence classes of admissible sequences $(x_i) \subset X$. It carries a natural Hausdorff topology with the property that the isometry group of X acts on ∂X as a group of homeomorphisms. For the complex of curves, the Gromov boundary was determined by Klarreich [K].

For the formulation of Klarreich's result, recall that a *geodesic lamination* for a complete hyperbolic structure of finite volume on S is a *compact* subset of S which is foliated into simple geodesics. A simple closed geodesic on S is a geodesic lamination with a single leaf. The space $\mathcal{L}$ of geodesic laminations on S can be equipped with the Hausdorff topology for compact subsets of S. With respect to this topology, $\mathcal{L}$ is compact and metrizable. A geodesic lamination is called *minimal* if each of its half-leaves is dense. A minimal geodesic lamination λ *fills up* S if every simple closed geodesic on S intersects λ transversely, i.e. if every complementary component of λ is an ideal polygon or a once punctured ideal polygon with geodesic boundary [CEG].

A geodesic lamination is *maximal* if its complementary regions are all ideal triangles or once punctured monogons. Note that a geodesic lamination can be both minimal and maximal (this unfortunate terminology is by now standard in the literature). Each geodesic lamination λ is a *sublamination* of a maximal lamination, i.e. there is a maximal lamination which contains λ as a closed subset [CEG]. For any minimal geodesic lamination λ which fills up S, the number of geodesic laminations μ which contain λ as a sublamination is bounded by a universal constant only depending on the topological type of the surface S. Namely, each such lamination μ can be obtained from λ by successively subdividing complementary components P of λ which are different from an ideal triangle or a once punctured monogon by adding a simple geodesic line which either connects two non-adjacent cusps of P or goes around a puncture in the interior of P. Notice that every leaf of μ which is not contained in λ is necessarily isolated in μ.

We say that a sequence $(\lambda_i) \subset \mathcal{L}$ *converges in the coarse Hausdorff topology* to a minimal lamination μ which fills up S if every accumulation point of (λ_i) with respect to the Hausdorff topology contains μ as a sublamination. We equip the space $\mathcal{B}$ of minimal geodesic laminations which fill up S with the following topology. A set $A \subset \mathcal{B}$ is closed if and only if for every sequence $(\lambda_i) \subset A$ which converges in the coarse Hausdorff topology to a lamination $\lambda \in \mathcal{B}$ we have $\lambda \in A$. We call this topology on $\mathcal{B}$ the *coarse Hausdorff topology*. Using this terminology, Klarreich's result [K] can be formulated as follows.

Theorem:

(1) *There is a natural homeomorphism Λ of $\mathcal{B}$ equipped with the coarse Hausdorff topology onto the Gromov boundary $\partial \mathcal{C}(S)$ of the complex of curves $\mathcal{C}(S)$ for S.*

(2) *For $\mu \in \mathcal{B}$ a sequence $(c_i) \subset \mathcal{C}(S)$ is admissible and defines the point $\Lambda(\mu) \in \partial \mathcal{C}(S)$ if and only if (c_i) converges in the coarse Hausdorff topology to μ.*

In the paper [K], Klarreich formulates her result using *measured foliations* on the surface S, i.e. topological foliations F on S equipped with a *transverse translation invariant measure*. The space $\mathcal{MF}$ of measured foliations can be equipped with the weak*-topology which is metrizable and hence Hausdorff. This topology projects to a metrizable topology on the space $\mathcal{PMF}$ of projective measured foliations which is the quotient of $\mathcal{MF}$ under the natural action of the positive half-line $(0, \infty)$. A topological foliation on S is called *minimal* if it does not contain a trajectory which is a simple closed curve. For every minimal topological foliation F, the set of projective measured foliations whose support equals F is a closed subset of $\mathcal{PMF}$. It follows that the quotient $\mathcal{Q}$ of the space of minimal projective measured foliations under the measure forgetting equivalence relation is a Hausdorff space as well. Note that the extended mapping class group of S acts on $\mathcal{Q}$ as a group of homeomorphisms. Klarreich shows that $\mathcal{Q}$ can be identified with the Gromov boundary of the complex of curves.

There is a natural map ι which assigns to a measured foliation F on S a *measured geodesic lamination* $\iota(F)$, i.e. a geodesic lamination λ together with a transverse translation invariant measure supported in λ. The geodesic lamination λ is the closure of the set of geodesics which are obtained by straightening the non-singular trajectories of the foliation (see [L] for details), together with the natural image of the transverse measure. A measured geodesic lamination can be viewed as a locally finite Borel measure on the space of unoriented geodesics in the hyperbolic plane which is invariant under the action of the fundamental group of S. Thus the space $\mathcal{ML}$ of measured geodesic laminations on S can be equipped with the restriction of the weak*-topology on the space of all such measures. With respect to this topology, the map ι is a homeomorphism of $\mathcal{MF}$ onto $\mathcal{ML}$ which factors to a homeomorphism of the space $\mathcal{PMF}$ of projective measured foliations onto the space $\mathcal{PML}$ of *projective measured laminations*, i.e. the quotient of $\mathcal{ML}$ under the natural action of $(0, \infty)$. This homeomorphism maps the space of minimal projective measured foliations onto the space $\mathcal{MPML}$ of projective measured geodesic laminations whose support is a minimal geodesic lamination which fills up S. Since every minimal geodesic lamination is the support of a transverse translation invariant measure (compare the expository article [Bo] for a discussion of this fact and related results), the image of $\mathcal{MPML}$ under the natural forgetful map Π which assigns to a projective measured geodesic lamination its support equals the set $\mathcal{B}$. As a consequence, our above theorem is just a reformulation of the result of Klarreich provided that the coarse Hausdorff topology on $\mathcal{B}$ is induced from the weak*-topology on $\mathcal{MPML}$ via the surjective map Π.

For this it suffices to show that the map Π is continuous and closed. To show continuity, let $(\mu_i) \subset \mathcal{MPML}$ be a sequence of projective measured geodesic laminations. Assume that $\mu_i \to \mu \in \mathcal{MPML}$ in the weak*-topology, so that the support $\Pi(\mu)$ of μ is contained in $\mathcal{B}$. Since the space of geodesic laminations equipped with the Hausdorff topology is compact, up to passing to a subsequence we may assume that the laminations $\Pi(\mu_i) \in \mathcal{B}$ converge as $i \to \infty$ in the Hausdorff topology to a geodesic lamination $\tilde{\lambda}$. Then $\tilde{\lambda}$ necessarily contains the support $\Pi(\mu) \in \mathcal{B}$ of μ as a sublamination and therefore $\Pi(\mu_i) \to \Pi(\mu)$ in the coarse Hausdorff topology. Note however that $\tilde{\lambda}$ may contain isolated leaves which are not contained in the support

of μ [CEG]. Since $\mathcal{MPML}$ and $\mathcal{B}$ are Hausdorff spaces, this shows that the map Π is indeed continuous.

To show that the map Π is closed, let $A \subset \mathcal{MPML}$ be a closed set and let $(\mu_i) \subset A$ be a sequence with the property that $(\Pi(\mu_i)) \subset \mathcal{B}$ converges in the coarse Hausdorff topology to a lamination $\lambda \in \mathcal{B}$. Up to passing to a subsequence we may assume that the geodesic laminations $\Pi(\mu_i)$ converge in the usual Hausdorff topology to a lamination $\tilde{\lambda}$ containing λ as a sublamination. Since the space of projective measured laminations is compact, after passing to another subsequence we may assume that the projective measures μ_i converge in the weak*-topology to a projective measure μ. Then μ is necessarily supported in $\tilde{\lambda}$. Now λ fills up S by assumption and therefore every transverse measure on $\tilde{\lambda}$ is supported in λ. Thus we have $\mu \in \mathcal{MPML}$ and, in particular, $\mu \in A$ since $A \subset \mathcal{MPML}$ is closed. This shows that Π is closed and consequently our theorem is just the main result of [K].

Klarreich's proof of the above theorem relies on Teichmüller theory and the results of Masur and Minsky in [MM1]. In this note we give a more combinatorial proof which uses train tracks and a result of Bowditch [B]. We discuss the relation between the complex of train tracks and the complex of curves in Section 2. The proof of the theorem is completed in Section 3.

2. The train track complex

A *train track* on S is an embedded 1-complex $\tau \subset S$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at the endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. Through each switch there is a path of class C^1 which is embedded in τ and contains the switch in its interior. In particular, the branches which are incident on a fixed switch are divided into "incoming" and "outgoing" branches according to their inward pointing tangent at the switch. Each closed curve component of τ has a unique bivalent switch, and all other switches are at least trivalent. The complementary regions of the train track have negative Euler characteristic, which means that they are different from discs with $0, 1$ or 2 cusps at the boundary and different from annuli and once-punctured discs with no cusps at the boundary. We always identify train tracks which are isotopic. Train tracks were probably used for the first time by Williams [W] to study recurrence properties of dynamical systems (I am grateful to Greg McShane for pointing this reference out to me). They became widely known through the work of Thurston about the structure of the mapping class group. A detailed account on train tracks can be found in [PH] and [M].

A train track is called *generic* if all switches are at most trivalent. The train track τ is called *transversely recurrent* if every branch b of τ is intersected by an embedded simple closed curve $c = c(b) \subset S$ which intersects τ transversely and is such that $S - \tau - c$ does not contain an embedded *bigon*, i.e. a disc with two corners at the boundary.

Recall that a geodesic lamination for a complete hyperbolic structure of finite volume on S is a compact subset of S which is foliated into simple geodesics. Particular geodesic laminations are simple closed geodesics, i.e. laminations which

consist of a single leaf. A geodesic lamination λ is called minimal if each of its half-leaves is dense in λ. Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves. Every geodesic lamination λ is a disjoint union of finitely many minimal components and a finite number of non-compact isolated leaves. An isolated leaf of λ either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components ([CEG], [O]).

A geodesic lamination λ is *maximal* if all its complementary components are ideal triangles or once punctured monogons. A geodesic lamination is called *complete* if it is maximal and can be approximated in the Hausdorff topology by simple closed geodesics. Every minimal geodesic lamination is a sublamination of a complete geodesic lamination [H]. The space $\mathcal{C}L$ of complete geodesic laminations on S equipped with the Hausdorff topology is compact.

A geodesic lamination or a train track λ is *carried* by a transversely recurrent train track τ if there is a map $F : S \to S$ of class C^1 which is isotopic to the identity and maps λ to τ in such a way that the restriction of its differential dF to every tangent line of λ is non-singular. Note that this makes sense since a train track has a tangent line everywhere. A train track τ is called *complete* if it is generic and transversely recurrent and if it carries a complete geodesic lamination [H].

A half-branch $\tilde{b}$ in a generic train track τ incident on a switch v is called *large* if the switch v is trivalent and if every arc $\rho : (-\epsilon, \epsilon) \to \tau$ of class C^1 which passes through v meets the interior of $\tilde{b}$. A branch b in τ is called *large* if each of its two half-branches is large; in this case b is necessarily incident on two distinct switches (for all this, see [PH]).

There is a simple way to modify a complete train track τ to another complete train track. Namely, if e is a large branch of τ then we can perform a right or left *split* of τ at e as shown in Figure A below. The split τ' of a train track τ is carried by τ. If τ is complete and if $\lambda \in \mathcal{C}L$ is carried by τ, then for every large branch e of τ there is a unique choice of a right or left split of τ at e with the property that the split track τ' carries λ, and τ' is complete. In particular, a complete train track τ can always be split at any large branch e to a complete train track τ'; however there may be a choice of a right or left split at e such that the resulting track is not complete any more (compare p.120 in [PH]).

Figure A

a   e   d

b   c

right split

left split
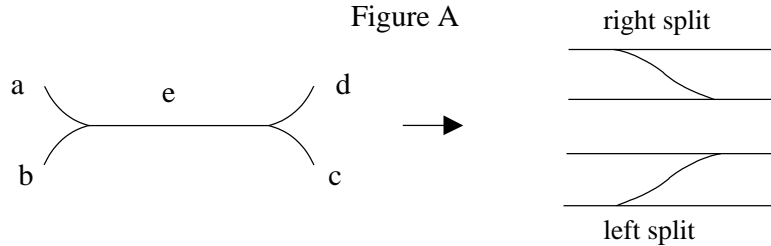
Let $\mathcal{T}T$ be the set of all isotopy classes of complete train tracks on S. We connect two train tracks τ, τ' with a directed edge if τ' can be obtained from τ by a single split at a large branch e. This provides $\mathcal{T}T$ with the structure of a locally finite

directed metric graph. The *mapping class group* $\mathcal{M}_{g,m}$ of all isotopy classes of orientation preserving homeomorphisms of S acts naturally on $\mathcal{T}T$ as a group of simplicial isometries. The following result is shown in [H].

Theorem 2.1. *The train track complex $\mathcal{T}T$ is connected, and the action of the mapping class group on $\mathcal{T}T$ is proper and cocompact.*

A *transverse measure* on a train track τ is a nonnegative weight function μ on the branches of τ satisfying the *switch condition*: For every switch s of τ, the sum of the weights over all incoming branches at s is required to coincide with the sum of the weights over all outgoing branches at s. The set $V(\tau)$ of all transverse measures on τ is a closed convex cone in a linear space and hence topologically it is a closed cell. The train track is called *recurrent* if it admits a transverse measure which is positive on every branch. A complete train track τ is recurrent [H].

A transverse measure μ on τ is called a *vertex cycle* [MM1] if μ spans an extreme ray in $V(\tau)$. Up to scaling, every vertex cycle μ is a *counting measure* of a simple closed curve c which is carried by τ. This means that for a carrying map $F : c \to \tau$ and every open branch b of τ the μ-weight of τ equals the number of connected components of $F^{-1}(b)$. More generally, every integral transverse measure μ for τ defines uniquely a simple *weighted geodesic multicurve*, i.e. there are simple closed pairwise disjoint geodesics $c_1, \dots, c_\ell$ and a carrying map $F : \cup_i c_i \to \tau$ such that $\mu = \sum a_i \nu_i$ where $a_i > 0$ is a positive integer and where ν_i is the counting measure for c_i. We have.

Lemma 2.2. *Let c be a simple closed curve which is carried by τ, with carrying map $F : c \to \tau$. Then c defines a vertex cycle on τ if and only if $F(c)$ passes through every branch of τ at most twice, with different orientation.*

Proof: Let $F : c \to \tau$ be a carrying map for a simple closed curve $c : S^1 \to S$ which defines a vertex cycle μ for τ. Assume to the contrary that there is a branch b of τ with the property that Fc passes through b twice in the same direction. Then there is a closed nontrivial subarc $[p,q] \subset S^1$ with nontrivial complement such that $F \circ c[p,q]$ and $F \circ c[q,p]$ are closed (not necessarily simple) curves on τ. For a branch e of τ define $\nu(e)$ to be the number of components of $(F \circ c[p,q])^{-1}(e)$. Then ν is a nontrivial nonnegative integral weight function on the branches of τ which clearly satisfies the switch condition, and the same is true for $\mu - \nu$. As a consequence, the transverse measure μ can be decomposed into a nontrivial sum of integral transverse measures which contradicts our assumption that μ is a vertex cycle for τ. This shows the first part of the lemma, and the second part follows in the same way. □

In the sequel we mean by a vertex cycle of a complete train track τ an *integral* transverse measure on τ which is the counting measure of a simple closed curve c on S carried by τ and which spans an extreme ray of $V(\tau)$; we also use the notion vertex cycle for the simple closed curve c. As a consequence of Lemma 2.2 and the fact that the number of branches of a complete train track on S only depends on

the topological type of S, the number of vertex cycles for a complete train track on S is bounded by a universal constant (see [MM1]).

Recall that the *intersection number* $i(\gamma, \delta)$ between two simple closed geodesics γ, δ equals the minimal number of intersection points between representatives of the free homotopy classes of γ, δ. This intersection number extends bilinearly to a pairing for weighted simple geodesic multicurves on S. The following corollary is immediate from Lemma 2.2. For its formulation, for a transverse measure μ on a train track τ denote by $\mu(\tau)$ the *total mass of μ*, i.e. $\mu(\tau) = \sum_b \mu(b)$ where b runs through the branches of τ. We have.

Corollary 2.3. *Let $\mu \in V(\tau)$ be an integral transverse measure on τ which defines the weighted simple geodesic multicurve c. Let ξ be any vertex cycle of τ; then $i(c, \xi) \leq 2\mu(\tau)$.*

Proof: Let c be any simple closed curve which is carried by the complete train track τ and denote by μ the counting measure on τ defined by c. Write $n = \mu(\tau)$; then there is a *trainpath* of length n, i.e. a C^1-immersion $\rho : [0, n] \to \tau$ which maps each interval $[i, i+1]$ onto a branch of τ and which parametrizes the image of c under a carrying map $c \to \tau$. We then can deform ρ with a smooth homotopy to a closed curve $\rho' : [0, n] \to S$ which is mapped to ρ by a carrying map and is such that for each $i \leq n$, $\rho'[i, i+1]$ intersects τ in at most one point contained in the interior of the branch $\rho[i, i+1]$.

Now let ξ be any vertex cycle of τ. By Lemma 2.2, ξ can be parametrized as a trainpath $\sigma : [0, s] \to \tau$ which passes through every branch of τ at most twice. Then the number of intersection points between σ and ρ' is not bigger than $2n = 2\mu(\tau)$. This shows the corollary for simple closed curves c which are carried by τ. The case of a general weighted simple geodesic multicurve carried by τ then follows from linearity of counting measures and the intersection form. $\square$

Since the distance in $\mathcal{C}(S)$ between two simple closed curves a, c is bounded from above by $2i(a, c) + 1$ [MM1], we obtain from Lemma 2.2 and Corollary 2.3 the existence of a number $D > 0$ with the property that for every train track $\tau \in \mathcal{TT}$, the distance in $\mathcal{C}(S)$ between any two vertex cycles of τ is at most D.

Define a map $\Phi : \mathcal{TT} \to \mathcal{C}(S)$ by assigning to a train track $\tau \in \mathcal{TT}$ a vertex cycle $\Phi(\tau)$ for τ. Every such map is roughly $\mathcal{M}_{g,m}$-equivariant. Namely, for $\psi \in \mathcal{M}_{g,m}$ and $\tau \in \mathcal{TT}$, the distance between $\Phi(\psi(\tau))$ and $\psi(\Phi(\tau))$ is at most D. Denote by d both the distance on $\mathcal{TT}$ and on $\mathcal{C}(S)$. We have.

Corollary 2.4. *There is a number $C > 0$ such that $d(\Phi(\tau), \Phi(\eta)) \leq Cd(\tau, \eta)$ for all $\tau, \eta \in \mathcal{TT}$.*

Proof: Let $\alpha : [0, m] \to \mathcal{TT}$ be any (simplicial) geodesic. Then for each i, either the train track $\alpha(i+1)$ is obtained from $\alpha(i)$ by a single split or $\alpha(i)$ is obtained from $\alpha(i+1)$ by a single split. Assume that $\alpha(i+1)$ is obtained from $\alpha(i)$ by a single split. Then there is a natural carrying map $F : \alpha(i+1) \to \alpha(i)$. By Lemma 2.2 and the definition of a split, via this carrying map the counting measure of a

vertex cycle c on $\alpha(i+1)$ defines an integral transverse measure on $\alpha(i)$ whose total mass is bounded from above by a universal constant. Thus by Corollary 2.3, the intersection number between c and any vertex cycle of $\alpha(i)$ is bounded from above by a universal constant. Then the distance in $\mathcal{C}(S)$ between c and any vertex cycle on $\alpha(i)$ is uniformly bounded as well [MM1]. This shows the corollary.   □

Define a *splitting sequence* in $\mathcal{T}T$ to be a (simplicial) map $\alpha : [0,m] \to \mathcal{T}T$ with the property that for each i the train track $\alpha(i+1)$ can be obtained from $\alpha(i)$ by a single split.

We use now a construction of Bowditch [B]. Recall the definition of the intersection form i on simple geodesic multicurves. For simple geodesic multicurves α, β on S with $i(\alpha, \beta) > 0$ and $a > 0, r > 0$ define

$$L_a(\alpha, \beta, r) = \{\gamma \in \mathcal{C}(S) \mid \max\{ai(\gamma, \alpha), i(\gamma, \beta)/ai(\alpha, \beta)\} \leq r\}.$$

Our next goal is to link the sets $L_a(\alpha, \beta, r)$ to splitting sequences. For this recall that a *pants decomposition* of S is a collection of $3g-3+m$ pairwise disjoint mutually not freely homotopic simple closed essential curves on S, i.e. these curves are not contractible and not freely homotopic into a puncture. Let $P = \{\gamma_1, \ldots, \gamma_{3g-3+m}\}$ be a pants decomposition for S. Then there is a special family of complete train tracks with the property that each pants curve γ_i admits a closed neighborhood A diffeomorphic to an annulus and such that $\tau \cap A$ is diffeomorphic to a *standard twist connector* depicted in Figure B. Such a train track clearly carries each pants curve from the pants decomposition P; we call it *adapted* to P (see [PH]). The set of train tracks adapted to a pants decomposition P is invariant under the action of $\mathcal{M}_{g,m}$. We show.

<div align="center">Figure B</div>
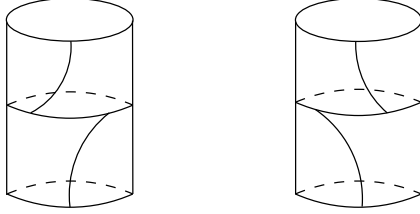
Lemma 2.5: *There is a number $k \geq 1$ with the following property. Let $\tau_0 \in \mathcal{T}T$ be adapted to a pants decomposition P of S, let $(\tau_i)_{0 \leq i \leq m} \subset \mathcal{T}T$ be a splitting sequence issuing from τ_0 and let α be a simple multicurve consisting of vertex cycles for τ_m. Then there is a monotonous surjective function $\kappa : (0, \infty) \to \{0, \ldots, m\}$ such that $\kappa(s) = 0$ for all sufficiently small $s > 0, \kappa(s) = m$ for all sufficiently large $s > 0$ and that for all $s \in (0, \infty)$ there is a vertex cycle of $\tau_{\kappa(s)}$ which is contained in $L_s(\alpha, P, k)$.*

Proof: Let P be a pants decomposition for S and let β be an arbitrary simple multicurve on S. Let $k > 1$ and assume that there is a curve $\gamma \in \mathcal{C}(S)$ with the property that $0 < c = i(P, \gamma)i(\gamma, \beta) \leq ki(P, \beta)$. Write $b = i(P, \gamma)/i(P, \beta), a =$

$i(\gamma, \beta)/c$; then $abi(P, \beta) = 1$ and $\max\{ai(P, \gamma), bi(\beta, \gamma)\} \leq k$. As a consequence, we have $\gamma \in L(P, \beta, k)$. Thus for the proof of our lemma we only have to show the existence of a number $k > 0$ with the following property. Let ζ be a train track which is adapted to a pants decomposition $P = \{\gamma_1, \ldots, \gamma_{3g-3+m}\}$ and let $\zeta : [0, m] \to \mathcal{TT}$ be a splitting sequence issuing from $\zeta(0) = \zeta$. Let $j > 0$ be such that the distance in $\mathcal{C}(S)$ between every vertex cycle of $\zeta(j)$ and every vertex cycle of $\zeta(0)$ is at least 3. Let ρ be a vertex cycle for $\zeta(m)$; then there is a vertex cycle $\alpha(j)$ for $\zeta(j)$ such that

$$(1) \qquad i(\rho, \alpha(j))\Big(\sum_{i=1}^{3g-3+m} i(\alpha(j), \gamma_i) \Big) \leq k \sum_{i=1}^{3g-3+m} i(\rho, \gamma_i).$$

Since $\zeta(0)$ is adapted to the pants decomposition P, every pants curve of P is a vertex cycle for $\zeta(0)$. Moreover, for each $i \leq 3g - 3 + m$ there is a branch b_i of $\zeta(0)$ contained in an annulus A_i about γ_i and such that the counting measure ν_i for γ_i is the *unique* vertex cycle of $\zeta(0)$ which gives positive mass to b_i. Thus the counting measure μ of any simple closed curve c which is carried by $\zeta(0)$ can be decomposed in a unique way as $\mu = \mu_0 + \sum_{i=1}^{3g-3+m} n_i\nu_i$ where $n_i \geq 0$ and where μ_0 is an integral transverse measure for $\zeta(0)$ with $\mu_0(b_i) = 0$ for all i. The intersection number of the curve c with a pants curve γ_i equals the μ_0-weight of the *large* branch e_i of $\zeta(0)$ contained in the annulus A_i. In particular, the intersection number of c and γ_i coincides with the intersection number of γ_i and the simple weighted multicurve c_0 defined by the transverse measure μ_0. Moreover, since the complement of P in S does not contain any essential closed curve which is not homotopic into a boundary component or a cusp, there is a constant k_0 only depending on the topological type of S with the property that

$$(2) \qquad \mu_0(\zeta(0)) \geq \sum_{i=1}^{3g-3+m} i(c, \gamma_i) = \sum_{i=1}^{3g-3+m} i(c_0, \gamma_i) \geq \mu_0(\zeta(0))/k_0.$$

Consider again the splitting sequence $\zeta : [0, m] \to \mathcal{TT}$ and let $j \leq m$ be such that the distance in $\mathcal{C}(S)$ between every vertex cycle of $\zeta(j)$ and every vertex cycle of $\zeta(0)$ is at least 3. Let ρ be a vertex cycle for the train track $\zeta(m)$. Since $\zeta(m)$ is carried by $\zeta(j)$, the curve ρ defines a counting measure η on $\zeta(j)$. This counting measure can (perhaps non-uniquely) be written in the form $\eta = \sum_{i=1}^{d} a_i\xi_i$ where ξ_i $(i = 1, \ldots, d)$ are the vertex cycles of $\zeta(j)$ and $a_i \geq 0$ are nonnegative integers. The number d of these vertex cycles is bounded from above by a universal constant and by Lemma 2.2, the total mass of each of these vertex cycles ξ_i is bounded from above by a universal constant as well. Therefore, there is a universal number $q > 0$ and there is some $i \leq d$ such that $a_i \geq \eta(\zeta(j))/q$. After reordering we may assume that $i = 1$. Write $\xi = \xi_1$; Corollary 2.3 shows that $i(\rho, \xi) \leq 2\eta(\zeta(j)) \leq 2qa_1$.

On the other hand, by our assumption on $\zeta(j)$ the distance in $\mathcal{C}(S)$ between ξ and each of the curves γ_i is at least 3. Thus ξ is mapped via the carrying map $\zeta(j) \to \zeta(0)$ to a curve in $\zeta(0)$ which together with each of the pants curves of P fills up S. Then ξ defines a counting measure μ on $\zeta(0)$, and we have $\mu = \mu_0 + \sum_{i=1}^{3g-3+m} p_i\nu_i$ for some $p_i \geq 0$ with $\mu_0 \neq 0$. By inequality (2), the sum of the intersection numbers between ξ and the curves γ_i is contained in the interval

$[\mu_0(\zeta(0))/k_0, \mu_0(\zeta(0))]$. On the other hand, by the choice of ξ and the fact that the carrying map $\zeta(j) \to \zeta(0)$ maps the convex cone $V(\zeta(j))$ of transverse measures on $\zeta(j)$ *linearly* into the convex cone of transverse measures on $\zeta(0)$, the counting measure for our curve ρ viewed as a curve which is carried by $\zeta(0)$ is of the form $a_1\mu + \mu'$; in particular, we have

$$(3) \quad \sum_i i(\rho, \gamma_i) \geq a_1\mu_0(\zeta(0))/k_0 \geq \eta(\zeta(j))\mu_0(\zeta(0))/qk_0 \geq i(\rho, \xi)\mu_0(\zeta(0))/2qk_0.$$

As a consequence of inequalities (2),(3) we have

$$(4) \qquad\qquad i(\rho, \xi)\sum_i i(\xi, \gamma_i) \leq 2qk_0 \sum_i i(\rho, \gamma_i).$$

This completes the proof of the lemma. $\square$

For any metric space (X, d) and any $L \geq 1$, a curve $\gamma : (a, b) \to X$ is called an *L-quasigeodesic* if for all $a < s < t < b$ we have

$$d(\gamma(s), \gamma(t))/L - L \leq t - s \leq Ld(\gamma(s), \gamma(t)) + L.$$

Since $\mathcal{C}(S)$ is a δ-hyperbolic geodesic metric space for some $\delta > 0$, every L-quasigeodesic of finite length is contained in a uniformly bounded neighborhood of a geodesic in $\mathcal{C}(S)$. Call a path $\gamma : [0, m] \to \mathcal{C}(S)$ an *unparametrized L-quasigeodesic* if there is some $s > 0$ and a homeomorphism $\sigma : [0, s] \to [0, m]$ such that the path $\gamma \circ \sigma : [0, s] \to \mathcal{C}(S)$ is an L-quasigeodesic. The image of every unparametrized L-quasigeodesic in $\mathcal{C}(S)$ of finite length is contained in a uniformly bounded neighborhood of a geodesic.

The following corollary is the key step toward the investigation of the Gromov boundary of $\mathcal{C}(S)$. It was first shown by Masur and Minsky [MM2], with a different proof.

Corollary 2.6. *There is a number $Q > 0$ such that the image under Φ of every splitting sequence in $\mathcal{TT}$ is an unparametrized Q-quasigeodesic.*

Proof: Recall the definition of the sets $L_a(\alpha, \beta, r)$ for $\alpha, \beta \in \mathcal{C}(S)$. Bowditch [B] showed that there is a number $r_0 > 0$ with the following property. Assume that $\alpha, \beta \in \mathcal{C}(S)$ *fill up* S, i.e. the distance $d(\alpha, \beta)$ between α and β in $\mathcal{C}(S)$ is at least 3; then we have.

(1) $L_a(\alpha, \beta, r_0) \neq \emptyset$ for all $a > 0$.
(2) For every $r > 0$, $a > 0$ the diameter of $L_a(\alpha, \beta, r)$ is bounded from above by a universal constant only depending on r.
(3) For $r > r_0$ there is a constant $q(r) > 0$ with the following property. For $a > 0$ choose some $\gamma(a) \in L_a(\alpha, \beta, r)$; then $\gamma : (0, \infty) \to \mathcal{C}(S)$ is an unparametrized $q(r)$-quasigeodesic with $d(\gamma(s), \alpha) \leq q(r)$ for all sufficiently large $s > 0$ and $d(\gamma(s), \beta) \leq q(r)$ for all sufficiently small $s > 0$.

Let again $\alpha, \beta \in \mathcal{C}(S)$ be such that α, β fill up S. For $r > r_0$ define

$$L(\alpha, \beta, r) = \cup_a L_a(\alpha, \beta, r).$$

By property 3) above and hyperbolicity of the complex of curves, there is a number $D(r) > 0$ only depending on r such that $L(\alpha, \beta, r)$ is contained in a tubular neighborhood of radius $D(r)$ about a geodesic connecting α to β.

Now let P be any pants decomposition for S containing the curve α and assume that $\gamma \in L_a(P, \beta, r)$ for some $r > 0$. Let α' be a pants curve of P so that $i(\alpha', \beta) = \max\{i(\nu, \beta) \mid \nu \in P\}$; then we have $\gamma \in L_a(\alpha', \beta, (3g - 3 + m)r)$. As a consequence of this, hyperbolicity of $\mathcal{C}(S)$ and Lemma 2.5, the image under Φ of the splitting sequence ζ is contained in a uniformly bounded neighborhood of any geodesic in $\mathcal{C}(S)$ connecting α to β. Since this consideration applies to *every* splitting sequence, "backtracking" of the assignment $j \to \Phi(\zeta(j))$ is excluded. From this the lemma is immediate. □

Remark: More generally, the proof of Corollary 2.6 also shows the following. Let $\zeta, \eta \in \mathcal{TT}$ and assume that η is carried by ζ. Let c be any simple closed curve which is carried by η; then $\Phi(\eta)$ is contained in a uniformly bounded neighborhood of a geodesic arc in $\mathcal{C}(S)$ connecting $\Phi(\zeta)$ to c.

3. PROOF OF THE THEOREM

Fix again a complete hyperbolic metric on S of finite volume. Recall that a *measured geodesic lamination* on S is a geodesic lamination equipped with a transverse translation invariant measure. As in the introduction we equip the space $\mathcal{ML}$ of measured geodesic laminations with the restriction of the weak*-topology. The Dirac mass on any simple closed geodesic c on S defines a measured geodesic lamination. The intersection of weighted simple geodesic multicurves extends to a continuous symmetric bilinear form i on $\mathcal{ML}$ which is called the *intersection form*. The support of a measured geodesic lamination μ for S is minimal and fills up S if and only if $i(\mu, \nu) > 0$ for every measured geodesic lamination ν on S whose support does not coincide with the support of μ. The space $\mathcal{PML}$ of *projective measured laminations* on S is the quotient of $\mathcal{ML}$ under the natural action of the multiplicative group $(0, \infty)$; it is homeomorphic to a sphere of dimension $6g - 6 + 2m - 1$ [FLP], in particular, it is compact. The complex of curves naturally embeds into $\mathcal{PML}$ by assigning to a simple closed geodesic its projectivized transverse Dirac mass.

Projective measured geodesic laminations can be used to study infinite sequences in the complex of curves. Denote again by d the distance on $\mathcal{C}(S)$. We have.

Lemma 3.1. *Let $(c_i) \subset \mathcal{C}(S)$ be a sequence which converges in $\mathcal{PML}$ to a projective measured lamination whose support λ_0 is minimal and fills up S. Let $k > 0$ and assume that $a_i \in \mathcal{C}(S)$ is such that $d(a_i, c_i) \leq k$; then up to passing to a subsequence, the sequence (a_i) converges in $\mathcal{PML}$ to a projective measured geodesic lamination supported in λ_0.*

Proof: We use an argument of Luo as explained in the proof of Proposition 4.6 of [MM1]. Namely, choose a continuous section $\iota : \mathcal{PML} \to \mathcal{ML} - \{0\}$ of the projection $\mathcal{ML} - \{0\} \to \mathcal{PML}$. Then every simple closed geodesic c on S defines a measured geodesic lamination $\hat{c} \in \iota(\mathcal{PML})$. Let $(c_i) \subset \mathcal{C}(S)$ be a sequence of simple closed geodesics. Assume that the sequence $(\hat{c}_i)$ converges in $\iota(\mathcal{PML})$ to a measured geodesic lamination ν_0 whose support λ_0 is minimal and fills up S.

Let $(a_i) \subset \mathcal{C}(S)$ be a sequence with $d(a_i, c_i) \leq k$ for a fixed number $k > 0$. By passing to a subsequence we may assume that $d(c_i, a_i)$ is independent of i, i.e. we may assume that $d(c_i, a_i) = k$ for all i. Then for each i there is a curve $c_i^1 \in \mathcal{C}(S)$ which is disjoint from c_i and such that $d(c_i^1, a_i) = k - 1$. Up to passing to a subsequence, the sequence $(\hat{c}_i^1) \subset \iota(\mathcal{PML})$ converges weakly to a measured geodesic lamination $\nu_1 \in \iota(\mathcal{PML})$. Since $i(\hat{c}_i^1, \hat{c}_i) = 0$ for all i, by continuity of the intersection form we have $i(\nu_0, \nu_1) = 0$ and therefore ν_1 is supported in λ_0. Proceeding inductively we conclude that up to passing to a subsequence, the measured laminations $\hat{a}_i$ defined by the curves a_i converge in $\iota(\mathcal{PML})$ to a measured lamination which is supported in λ_0. This shows the lemma. $\qquad\square$

Consider again the train track complex $\mathcal{TT}$. For $\tau \in \mathcal{TT}$ denote by $A(\tau) \subset \mathcal{CL}$ the set of all complete geodesic laminations carried by τ. Then $A(\tau)$ is open and closed in $\mathcal{CL}$. Following [H], define a *full splitting sequence* in $\mathcal{TT}$ to be a sequence $\alpha : [0, \infty) \to \mathcal{TT}$ with the property that for every $i \geq 0$, the train track $\alpha(i+1)$ is obtained by splitting $\alpha(i)$ at each of the large branches precisely once. If $\tau \in \mathcal{TT}$ is arbitrary and if $\lambda \in \mathcal{CL}$ is a complete geodesic lamination which is carried by τ, then λ determines uniquely a full splitting sequence $\alpha_{\tau, \lambda}$ issuing from τ by requiring that each of the train tracks $\alpha_{\tau, \lambda}(i)$ carries λ, and $\cap_i A(\alpha_{\tau, \lambda}(i)) = \{\lambda\}$ [H]. Recall the definition of the map $\Phi : \mathcal{TT} \to \mathcal{C}(S)$. By Corollary 2.6, there is a universal number $Q > 0$ such that the curve $i \to \Phi(\alpha_{\tau, \lambda}(i))$ is an unparametrized Q-quasigeodesic in $\mathcal{C}(S)$. This means that this curve defines a quasiisometric embedding of the half-line $[0, \infty)$ into $\mathcal{C}(S)$ if and only if the diameter in $\mathcal{C}(S)$ of the set $\Phi(\alpha_{\tau, \lambda}[0, \infty))$ is infinite.

Let $\mathcal{B}$ be the set of all minimal geodesic laminations on S which fill up S, equipped with the coarse Hausdorff topology. Recall that $\mathcal{B}$ is a Hausdorff space. The next statement is immediate from Lemma 3.1.

Corollary 3.2: *Let $\lambda \in \mathcal{CL}$ be a complete geodesic lamination which contains a sublamination $\lambda_0 \in \mathcal{B}$. Let $\tau \in \mathcal{TT}$ be a train track which carries λ; then the diameter of the set $\Phi(\alpha_{\tau, \lambda}[0, \infty)) \subset \mathcal{C}(S)$ is infinite.*

Proof: Let $\lambda \in \mathcal{CL}$ be a complete geodesic lamination which contains a sublamination $\lambda_0 \in \mathcal{B}$. Assume that λ is carried by a train track $\tau \in \mathcal{TT}$. Denote by $\alpha_\lambda = \alpha_{\tau, \lambda}$ the full splitting sequence issuing from τ which is determined by λ. We have to show that the diameter of the set $\Phi(\alpha_\lambda[0, \infty))$ is infinite. For this recall that $\cap_i A(\alpha_\lambda(i)) = \{\lambda\}$. Since for each i the curve $\Phi(\alpha_\lambda(i))$ is carried by $\alpha_\lambda(i)$, the curves $\Phi(\alpha_\lambda(i))$ viewed as projective measured laminations converge up to passing to a subsequence as $i \to \infty$ in $\mathcal{PML}$ to a projective measured geodesic lamination which is supported in λ_0. Thus by Lemma 3.1, there is no curve $a \in \mathcal{C}(S)$ with

$d(\Phi(\alpha_\lambda(i)), a) \leq k$ for a fixed number $k > 0$ and all i and hence the diameter in $\mathcal{C}(S)$ of the set $\Phi(\alpha_\lambda[0, \infty))$ is indeed infinite. $\square$

As in the introduction, we call a sequence $(c_i) \subset \mathcal{C}(S)$ admissible if for a fixed $p \in \mathcal{C}(S)$ we have $(c_i, c_j)_p \to \infty$ $(i, j \to \infty)$. Two admissible sequences $(a_i), (c_i) \subset \mathcal{C}(S)$ are equivalent if $(a_i, c_i)_p \to \infty (i \to \infty)$. The Gromov boundary $\partial \mathcal{C}(S)$ of $\mathcal{C}(S)$ is the set of equivalence classes of admissible sequences in $\mathcal{C}(S)$. Note that any quasigeodesic ray in $\mathcal{C}(S)$ defines an admissible sequence. We use Corollary 3.2 to show.

Lemma 3.3. *There is an injective map* $\Lambda : \mathcal{B} \to \partial \mathcal{C}(S)$.

Proof: Fix a pants decomposition P of S. Then there is a finite collection $\tau_1, \ldots, \tau_\ell \subset \mathcal{TT}$ of train tracks adapted to P with the property that every complete geodesic lamination $\lambda \in \mathcal{CL}$ is carried by one of the tracks τ_i (see [PH], [H]). Let $\mathcal{A} \subset \mathcal{CL}$ be the set of all complete geodesic laminations which contain a sublamination $\lambda_0 \in \mathcal{B}$. For $\lambda \in \mathcal{A}$ let τ_j be a train track from our collection $\tau_1, \ldots, \tau_\ell$ which carries λ and let $\alpha_\lambda : [0, \infty) \to \mathcal{TT}$ be the full splitting sequence issuing from τ_j which is determined by λ. By Corollary 2.6 and Corollary 3.2, there is a universal number $Q > 0$ with the property that the curve $i \to \Phi(\alpha_\lambda(i))$ is an unparametrized Q-quasigeodesic of infinite diameter. Hence this curve defines a point $\tilde{\Lambda}(\lambda) \in \partial \mathcal{C}(S)$.

There is a natural continuous projection $\pi : \mathcal{A} \to \mathcal{B}$ which maps a lamination $\lambda \in \mathcal{A}$ to its unique minimal sublamination $\pi(\lambda) \in \mathcal{B}$. We claim that $\tilde{\Lambda}(\lambda) = \tilde{\Lambda}(\mu)$ for $\lambda, \mu \in \mathcal{A}$ if $\pi(\lambda) = \pi(\mu) = \lambda_0$. For this extend the map Φ to the collection of all recurrent train tracks on S by assigning to such a train track σ a vertex cycle $\Phi(\sigma)$ of σ. Since the minimal sublamination $\lambda_0 = \pi(\lambda)$ of λ fills up S and is carried by each of the train tracks $\alpha_\lambda(i)$, the image of λ_0 under a carrying map $\lambda \to \alpha_\lambda(i)$ is a *recurrent* subtrack $\hat{\alpha}_\lambda(i)$ of $\alpha_\lambda(i)$ which is *large*. This means that $\hat{\alpha}_\lambda(i)$ is a train track on S which is a subset of $\alpha_\lambda(i)$ and whose complementary components do not contains an essential simple closed curve which is not homotopic into a puncture. By Lemma 2.2, every vertex cycle for $\hat{\alpha}_\lambda(i)$ is also a vertex cycle for $\alpha_\lambda(i)$ and therefore the distance between $\Phi(\alpha_\lambda(i))$ and $\Phi(\hat{\alpha}_\lambda(i))$ is bounded by a universal constant.

Up to isotopy, the train tracks $\hat{\alpha}_\lambda(i)$ converge as $i \to \infty$ in the Hausdorff topology to the lamination λ_0 (see [M],[H]). Since λ_0 is a sublamination of μ, for every $i > 0$ there is a number $j(i) > 0$ such that the train track $\hat{\alpha}_\lambda(j(i))$ is carried by $\alpha_\mu(i)$ (see [H]). By the remark following the proof of Corollary 2.6, this implies that $\Phi(\alpha_\mu(i))$ is contained in a uniformly bounded neighborhood of $\Phi(\alpha_\lambda[0, \infty))$. Since $i \geq i_0$ was arbitrary, the Hausdorff distance between the Q-quasigeodesics in $\mathcal{C}(S)$ defined by λ, μ is bounded and hence we have $\tilde{\Lambda}(\lambda) = \tilde{\Lambda}(\mu)$ as claimed. Thus there is a map $\Lambda : \mathcal{B} \to \partial \mathcal{C}(S)$ such that $\tilde{\Lambda} = \Lambda \circ \pi$.

We claim that the map Λ is injective. For this let $\lambda_0 \neq \mu_0 \in \mathcal{B}$ and let $\lambda \in \pi^{-1}(\lambda_0) \subset \mathcal{A}, \mu \in \pi^{-1}(\mu_0) \subset \mathcal{A}$. By Corollary 2.6 and Corollary 3.2, the image under Φ of full splitting sequences $\alpha_\lambda, \alpha_\mu \in \mathcal{TT}$ determined by λ, μ are

unparametrized Q-quasigeodesics in $\mathcal{C}(S)$ of infinite diameter. Thus by the definition of Λ, we have $\Lambda(\lambda_0) = \Lambda(\mu_0)$ if and only if the Hausdorff distance between $\Phi(\alpha_\lambda[0,\infty))$ and $\Phi(\alpha_\mu[0,\infty))$ is finite.

Assume to the contrary that this is the case. Then there is a number $D > 0$ and for every $i > 0$ there is a number $j(i) > 0$ such that $d(\Phi(\alpha_\lambda(i)), \Phi(\alpha_\mu(j(i)))) \leq D$. Since $d(\Phi(\alpha_\lambda(0)), \Phi(\alpha_\lambda(i))) \to \infty$ we have $j(i) \to \infty$ $(i \to \infty)$ by Corollary 2.4. Therefore, up to passing to a subsequence, the curves $\Phi(\alpha_\lambda(i))$, $\Phi(\alpha_\mu(j(i)))$ viewed as projective measured geodesic laminations converge as $i \to \infty$ to projective measured geodesic laminations ν_0, ν_1 supported in λ_0, μ_0. But λ_0, μ_0 fill up S and do not coincide and hence this contradicts Lemma 3.1. $\qquad\square$

The Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ admits a natural Hausdorff topology which can be described as follows. Extend the Gromov product $(\,,\,)_p$ to a product on $\partial\mathcal{C}(S)$ by defining $(\xi, \zeta)_p = \sup \liminf_{i,j\to\infty}(x_i, y_j)_p$ where the supremum is taken over all admissible sequences $(x_i), (y_j)$ representing the points ξ, ζ. We have $(\xi, \zeta)_p = \infty$ if and only if $\xi = \zeta \in \partial\mathcal{C}(S)$. A subset U of $\partial\mathcal{C}(S)$ is a neighborhood of a point $\xi \in \partial\mathcal{C}(S)$ if and only if there is a number $\epsilon > 0$ such that $\{\zeta \in \partial\mathcal{C}(S) \mid e^{-(\xi,\zeta)_p} < \epsilon\} \subset U$ (compare [BH]).

We say that a sequence $(c_i) \subset \mathcal{C}(S)$ *converges in the coarse Hausdorff topology* to a lamination $\mu \in \mathcal{B}$ if every accumulation point of (c_i) with respect to the Hausdorff topology contains μ as a sublamination. The next lemma completes the proof of our theorem from the introduction.

Lemma 3.4:

(1) *The map $\Lambda : \mathcal{B} \to \partial\mathcal{C}(S)$ is a homeomorphism.*
(2) *For $\mu \in \mathcal{B}$, a sequence $(c_i) \subset \mathcal{C}(S)$ is admissible and defines the point $\Lambda(\mu) \in \partial\mathcal{C}(S)$ if and only if $c_i \to \mu$ in the coarse Hausdorff topology.*

Proof: We show first the following. Let $(c_i) \subset \mathcal{C}(S)$ be an admissible sequence, i.e. a sequence with the property that $(c_i, c_j)_p \to \infty$ $(i, j \to \infty)$. Then there is some $\lambda_0 \in \mathcal{B}$ such that (c_i) converges in the coarse Hausdorff topology to λ_0.

For this we first claim that there is a number $b > 0$ and an admissible sequence $(a_j) \subset \mathcal{C}(S)$ which is equivalent to (c_i) (i.e. which satisfies $(a_i, c_i)_p \to \infty$) and such that the assignment $j \to a_j$ is a b-quasigeodesic in $\mathcal{C}(S)$.

Namely, let $j > 0$ and choose a number $n(j) > j$ such that $(c_\ell, c_n)_p \geq j$ for all $\ell, n \geq n(j)$. By hyperbolicity, this means that there is a point $a_j \in \mathcal{C}(S)$ with $d(p, a_j) \geq j$ and the property that for $n \geq n(j)$, every geodesic connecting c_n to p passes through a neighborhood of the point a_j of uniformly bounded diameter not depending on j. By construction, the sequence $(a_j) \subset \mathcal{C}(S)$ is contained in a b-quasigeodesic for a number $b > 0$ only depending on the hyperbolicity constant, and this quasigeodesic defines the same equivalence class as the sequence (c_j). As a consequence, we may assume without loss of generality that (c_i) is a uniform quasigeodesic. By the considerations in Section 2 we may moreover assume that there is a splitting sequence $(\tau_j)_{j\geq 0} \subset \mathcal{T}T$ and a strictly increasing function $\sigma :$

$\mathbb{N} \to \mathbb{N}$ such that $c_i = \Phi(\tau_{\sigma(i)})$ where $\Phi : \mathcal{TT} \to \mathcal{C}(S)$ assigns to a train track τ a vertex cycle for τ.

By Lemma 2.5 there is a number $k > 0$ with the property that for all $0 < i < j$ there is a vertex cycle $a_{i,j} \in \mathcal{C}(S)$ for $\tau_{\sigma(i)}$ such that

$$(5) \qquad\qquad i(c_0, a_{i,j})i(a_{i,j}, c_j) \leq ki(c_0, c_j) \quad \text{for } 0 < i < j.$$

Note that this inequality is invariant under multiplication of the simple closed curve $a_{i,j}$ with an arbitrary positive weight. Let again $\iota : \mathcal{PML} \to \mathcal{ML} - \{0\}$ be a continuous section and for $j > 0$ let $\hat{c}_j \in \iota(\mathcal{PML})$ be a multiple of c_j. By passing to a subsequence we may assume that the sequence $(\hat{c}_j)$ converges in the space of measured geodesic laminations to a measured geodesic lamination μ.

We claim that the support of μ is a minimal geodesic lamination which fills up S. For this we argue by contradiction and we assume otherwise. Then there is a simple closed curve c on S with $i(c, \mu) = 0$ (it is possible that the curve c is a minimal component of the support of μ). Replace the quasigeodesic (c_i) by an equivalent quasigeodesic, again denoted by (c_i), which issues from $c = c_0$ and which eventually coincides with the original quasigeodesic. Such a quasigeodesic exists by hyperbolicity of $\mathcal{C}(S)$. Since the number of vertex cycles for a fixed train track is bounded from above by a universal constant, after passing to a subsequence and using a standard diagonal argument we may assume that the curve $a_{i,j}$ is independent of $j > i$; we denote this curve by a_i. Inequality (5) and continuity of the intersection form then implies that $i(c, a_i)i(a_i, \hat{c}_j) \leq k_0 i(c, \mu) = 0$ for all $i > 0$. Since $d(c, a_i) \geq d(c, c_i) - k_0$ for all i, for $i > k_0 + 2$ the intersection numbers $i(c, a_i)$ are bounded from below by a universal constant and therefore $i(a_i, \mu) = 0$ for all $i > 0$. If the support of μ contains a simple closed curve component a, then this just means that the set $\{a_i \mid i > 0\} \subset \mathcal{C}(S)$ is contained in the $k_0 + 1$-neighborhood of a which is impossible. Otherwise μ has a minimal component μ_0 which fills a nontrivial bordered subsurface S_0 of S, and $i(\mu_0, a) > 0$ for every simple closed curve a in S which is contained in S_0 and which is not freely homotopic into a boundary component or a cusp. Since $i(a_i, \mu) = 0$ by assumption, the curves a_i do not have an essential intersection with S_0 which means that $i(a_i, a) = 0$ for every simple closed essential curve a in S_0. Again we deduce that the set $\{a_i \mid i > 0\} \subset \mathcal{C}(S)$ is bounded. Together we obtain a contradiction which implies that indeed the support of μ is a minimal geodesic lamination $\lambda_0 \in \mathcal{B}$ which fills up S.

Let λ_i be a complete geodesic lamination which contains c_i as a minimal component. By passing to a subsequence we may assume that the laminations λ_i converge in the Hausdorff topology to a complete geodesic lamination λ. Since the measured laminations $\hat{c}_j$ converge in the weak*-topology to μ, the lamination λ necessarily contains λ_0 as a sublamination.

Let α_λ be a full splitting sequence determined by λ. For every $i > 0$ the set of complete geodesic laminations which are carried by $\alpha_\lambda(i)$ is an open neighborhood of λ in $\mathcal{CL}$. Thus for every $i > 0$ there is a number $j(i) > 0$ with the property that for every $j \geq j(i)$ the geodesic c_j is carried by $\alpha_\lambda(i)$. From the remark following Corollary 2.6 we conclude that $\Phi(\alpha_\lambda(i))$ is contained in a uniformly bounded neighborhood of any geodesic connecting c_j to $\Phi(\alpha_\lambda(0))$. As a consequence, the image under Φ of the full splitting sequence α_λ defines the same point in the Gromov

boundary of $\mathcal{C}(S)$ as (c_j). In other words, the point in $\partial\mathcal{C}(S)$ defined by (c_j) equals $\Lambda(\lambda_0)$ and the map Λ is surjective. Hence by Lemma 3.3, the map Λ is a bijection. Moreover, if $(c_i) \subset \mathcal{C}(S)$ is any admissible sequence and if (c_{i_j}) is any subsequence with the property that the curves c_{i_j} converge in the Hausdorff topology to a geodesic lamination λ, then λ contains a lamination $\lambda_0 \in \mathcal{B}$ as a minimal component, and (c_i) defines the point $\Lambda(\lambda_0) \in \mathcal{C}(S)$. In particular, for every admissible sequence $(c_i) \subset \mathcal{C}(S)$ the curves (c_i) converge in the coarse Hausdorff topology to the lamination $\lambda_0 = \Lambda^{-1}((c_i)) \in \mathcal{B}$. This shows our above claim.

Let again $\mathcal{L}$ be the space of all geodesic laminations on S equipped with the Hausdorff topology. Let $\mathcal{A} \subset \mathcal{L}$ be the set of all laminations containing a minimal sublamination which fills up S. Above we defined a projection $\pi : \mathcal{A} \to \mathcal{B}$. Let $\lambda_0 \in \mathcal{B}$ and let $L = \pi^{-1}(\lambda_0) \subset \mathcal{A}$ be the set of all geodesic laminations which contain λ_0 as a sublamination. Since λ_0 fills up S, the set L is finite. We call a subset V of $\mathcal{C}(S) \cup \mathcal{B}$ a *neighborhood* of λ_0 in the *coarse Hausdorff topology of* $\mathcal{C}(S) \cup \mathcal{B}$ if there is a neighborhood W of L in $\mathcal{L}$ such that $V \supset (W \cap \mathcal{C}(S)) \cup \pi(W \cap \mathcal{A})$.

For $\xi \in \partial\mathcal{C}(S)$ and $c \in \mathcal{C}(S)$ write $(c, \xi)_p = \sup_{(x_i)} \liminf_{i \to \infty} (c, x_i)_p$ where the supremum is taken over all admissible sequences (x_i) defining ξ. A subset U of $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ is called a neighborhood of $\xi \in \partial\mathcal{C}(S)$ if there is some $\epsilon > 0$ such that U contains the set $\{\zeta \in \mathcal{C}(S) \cup \partial\mathcal{C}(S) \mid e^{-(\xi, \zeta)_p} < \epsilon\}$. In the sequel we identify $\mathcal{B}$ and $\partial\mathcal{C}(S)$ with the bijection Λ. In other words, we view a point in $\partial\mathcal{C}(S)$ as a minimal geodesic lamination which fills up S, i.e. we suppress the map Λ in our notation. To complete the proof of our lemma it is now enough to show the following. A subset U of $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$ is a neighborhood of $\lambda_0 \in \mathcal{B} = \partial\mathcal{C}(S)$ if and only if U is a neighborhood of λ_0 in the coarse Hausdorff topology.

For this let $\lambda_0 \in \mathcal{B}$, let $L = \pi^{-1}(\lambda_0) \subset \mathcal{A}$ be the collection of all geodesic laminations containing λ_0 as a sublamination and let $p = \Phi(\tau)$ for a train track $\tau \in \mathcal{TT}$ which carries each of the laminations $\lambda \in L$ (see [H] for the existence of such a train track τ). Let $\epsilon > 0$ and let $U = \{\zeta \in \mathcal{C}(S) \cup \partial\mathcal{C}(S) \mid e^{-(\lambda_0, \zeta)_p} < \epsilon\}$. Let $\lambda_1, \ldots, \lambda_s \subset L$ be the collection of all complete geodesic laminations contained in L and for $i \leq s$ let α_i be the full splitting sequence issuing from $\alpha_i(0) = \tau$ which is determined by λ_i. By hyperbolicity and the remark after Corollary 2.6, there is a universal constant $\chi > 0$ with the property that for each $i \leq s, j \geq 0$ every geodesic connecting p to a curve $c \in \mathcal{C}(S)$ which is carried by $\alpha_i(j)$ passes through the χ-neighborhood of $\Phi(\alpha_i(j))$. Since $\Phi(\alpha_i)$ is an unparametrized quasigeodesic which represents the point $\lambda_0 \in \partial\mathcal{C}(S)$, this implies that there is a number $j > 0$ such that $e^{-(c, \lambda_0)_p} < \epsilon$ and $e^{-(\mu, \lambda_0)_p} < \epsilon$ for all simple closed curves $c \in \mathcal{C}(S)$ and all laminations $\mu \in \mathcal{B}$ which are carried by one of the train tracks $\alpha_i(j)$ $(i = 1, \ldots, s)$. Since the set of all geodesic laminations which are carried by the train tracks $\alpha_i(j)$ $(i = 1, \ldots, s)$ is a neighborhood of L in $\mathcal{L}$ with respect to the Hausdorff topology (see [H]), we conclude that a neighborhood of λ_0 in $\partial\mathcal{C}(S) \cup \mathcal{C}(S)$ is a neighborhood of λ_0 in the coarse Hausdorff topology as well.

To show that a neighborhood of $\lambda_0 \in \mathcal{B}$ in the coarse Hausdorff topology contains a set of the form $\{\zeta \in \mathcal{C}(S) \cup \partial\mathcal{C}(S) \mid e^{-(\lambda_0, \zeta)_p} < \epsilon\}$ we argue by contradiction. Let again $L = \pi^{-1}(\lambda_0) \subset \mathcal{A}$ be the collection of all geodesic laminations containing λ_0 as a sublamination. Assume that there is an open neighborhood $W \subset \mathcal{L}$ of L in the

Hausdorff topology with the property that $\pi(W \cap \mathcal{A}) \cup (W \cap \mathcal{C}(S))$ does not contain a neighborhood of λ_0 in $\mathcal{C}(S) \cup \partial\mathcal{C}(S)$. Let $(c_i) \subset \mathcal{C}(S)$ be a sequence which represents $\lambda_0 \in \mathcal{B} = \partial\mathcal{C}(S)$. By our above consideration, every accumulation point of $(c_i) \subset \mathcal{L}$ with respect to the Hausdorff topology is contained in L. By our assumption, there is a sequence $i_j \to \infty$, a sequence $(a_j) \subset \mathcal{C}(S)$ and a sequence $R_j \to \infty$ such that $(c_{i_j}, a_j)_p \geq R_j$ and that $a_j \notin W$. By passing to a subsequence we may assume that the curves a_j converge in the Hausdorff topology to a lamination $\zeta \notin W$. However, since $(c_{i_j}, a_j)_p \to \infty$, the sequence (a_j) is admissible and equivalent to (c_j) and therefore by our above consideration, (a_j) converges in the coarse Hausdorff topology to λ_0. Then $a_j \in W$ for all sufficiently large j which is a contradiction. This shows our above claim and completes the proof of our lemma. $\qquad\square$

4. References

[Bo] F. Bonahon, *Geodesic laminations on surfaces*, in "Laminations and foliations in dynamics, geometry and topology" (Stony Brook, NY, 1998), 1–37, Contemp. Math., 269, Amer. Math. Soc., Providence, RI, 2001.

[B] B. Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, preprint 2002.

[BH] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Grundlehren 319, Springer, Berlin 1999.

[CEG] R. Canary, D. Epstein, P. Green, *Notes on notes of Thurston*, in "Analytical and geometric aspects of hyperbolic space", edited by D. Epstein, London Math. Soc. Lecture Notes 111, Cambridge University Press, Cambridge 1987.

[FLP] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque 1991.

[H] U. Hamenstädt, *Train tracks and mapping class groups I*, preprint 2004.

[Ha] W. J. Harvey, *Boundary structure of the modular group*, in "Riemann Surfaces and Related topics: Proceedings of the 1978 Stony Brook Conference" edited by I. Kra and B. Maskit, Ann. Math. Stud. 97, Princeton, 1981.

[I] N. V. Ivanov, *Mapping class groups*, Chapter 12 in "Handbook of Geometric Topology", edited by R.J. Daverman and R.B. Sher, Elsevier Science (2002), 523-633.

[K] E. Klarreich, *The boundary at infinity of the curve complex and the relative Teichmüller space,* preprint 1999.

[L] G. Levitt, *Foliations and laminations on hyperbolic surfaces,* Topology 22 (1983), 119-135.

[MM1] H. Masur, Y. Minsky, *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. 138 (1999), 103-149.

[MM2] H. Masur, Y. Minsky, *Quasiconvexity in the curve complex*, preprint 2003.

[M] L. Mosher, *Train track expansions of measured foliations*, unpublished manuscript.

[O] J. P. Otal, *Le Théorème d'hyperbolisation pour les variétés fibrées de dimension 3*, Astérisque 235, Soc. Math. Fr. 1996.

[PH] R. Penner with J. Harer, *Combinatorics of train tracks*, Ann. Math. Studies 125, Princeton University Press, Princeton 1992.

[W] R.F. Williams, *One-dimensional non-wandering sets*, Topology 6 (1967), 473-487.